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ON THE BENDING PROBLEMS OF ANISOTROPIC (ORTHOTROPIC) PLATES RESTING ON ELASTIC FOUNDATIONS THAT REACT IN COMPRESSION ONLY

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Abstract—In several works, the bending of plates resting on elastic foundations that react in compression only have been investigated. In all of these investigations, the bending of the plates is described within the framework of Kirchhoff–Love hypothesis and the material of those plates are supposed to be isotropic. Therefore, the results of the above-mentioned investigations are not suitable for the bending of the plates fabricated from the composite materials and resting on the elastic foundation that react in compression only.

In the present paper, the development of the solution method of the bending problems of the plates fabricated from the composite materials on the "tensionless" elastic foundation is proposed and the influence of the plate material properties to the displacement distribution and the form of the contact region is studied with concrete problems as an example. In this case, the plate material is modelled as an anisotropic material with the normalized mechanical properties; and the bending of the plate is described by using some version of the higher order refined plate theory. Moreover, with respect to the foundation, the following cases are considered; (1) Winkler foundation; (2) elastic half-space. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

At present, in numerous investigations, the bending of the plates resting on the elastic foundations have been studied. Among these investigations those which deal with the bending of the plates on the "tensionless" foundations are of considerable importance. In this field, one of the attempts has been made in the papers of Hussian *et al.* (1968) and Weitsman (1969), in which the foundation is represented by an elastic solid and the investigation of the considered problem has been reduced to the solution of the dual integral equations. Moreover, in the investigations of Gladwell and Iyer (1974) and Li and Dempsey (1988), the bending of circular and square plates on a "tensionless" elastic half-space has been studied. It is necessary to note that the considerable number of the investigations presented in the papers of Weitsman (1970), Celep (1988), Khathlan (1994) and others, have been made in the framework on the base of simply foundation models, which are detailed in the investigations of Kerr (1964) and Vlasov and Leontiev (1966).

Note that, the first attempt of accounting the anisotropy of the material of the plate resting on the Winkler foundation that reacts in compression only has been made in the paper of Kocatürk (1995). However, in this paper the behaviour of the plate bending is described in the framework of the Kirchhof-Love hypothesis. Therefore, the results of Kocatürk (1995) can not take into account the influence of the weak shear stiffness (which is one of the major specific characteristics of the composite materials) on the bending of the plate fabricated from the composite materials and resting on the foundation that reacts in compression only.

Taking into account the above-stated facts, in the present paper the development of the solution method of the bending problems of the plates fabricated from the composite materials on the "tensionless" elastic foundation is proposed by using some version of the higher order refined plate theories, with concrete problems as examples. In this case, the plate material is modelled as an anisotropic material with the normalized mechanical properties. Moreover, with respect to the foundation, the following cases are considered: (1) Winkler foundation; (2) elastic half-space.

2. FORMULATION OF THE PROBLEM AND BASIC EQUATIONS

Consider a rectangular plate with thickness h which rests on the elastic half-space and where contact between them is unbonded and frictionless. Associate the rectangular Cartesian system of coordinates $ox_1x_2x_3$ with the middle plane of this plate and suppose that the considered plate is located in plane ox_1x_2 and occupies the region $-a/2 \leqslant x_1 \leqslant a/2$, $-b/2 \leqslant x_2 \leqslant b/2$, $-h/2 \leqslant x_3 \leqslant h/2$ (see Fig. 1). Assume that, on the upper surface of the plate, the external forces and moments act. The material of the plate is also assumed to be homogeneous, orthotropic with the principal axes of elastic symmetry ox_1, ox_2 and ox_3.

Denoting the displacement of the middle plane in the directions ox_1, ox_2 and ox_3 by u_{10}, u_{20} and W, respectively, we suppose that the inequalities $u_{10} \ll W$, $u_{20} \ll W$ are satisfied and u_{10}, u_{20} are neglected in all discussions below.

In this paper, we use a higher-order refined theory for anisotropic plates presented in the papers of Kromm (1953, 1955) that is a special case of Gol'denveizer's (1958) extension of Reissnerr's theory (1945). Note that Kromm's theory (1953, 1955) is one of the versions of higher-order refined theories for anisotropic plates (the review of these theories for layered composites is given in detail in Reddy (1990)) and is more suitable for our aim in the sense of the generalization of the known results obtained for isotropic plates resting on the elastic half-space. Thus, within Kromm's theory (1953, 1955) and the framework of the above-stated, the displacement field of the plate considered may be written in the following form:

$$u_i = -x_3 \frac{\partial W(x_1, x_2)}{\partial x_i} + \frac{x_3}{2}\left(\frac{h^2}{4} - \frac{x_3^2}{3}\right)\varphi_i(x_1, x_2), \quad i = 1,2 \tag{1}$$

where φ_1, φ_2 characterize the average values of shear strains in the plate cross sections that

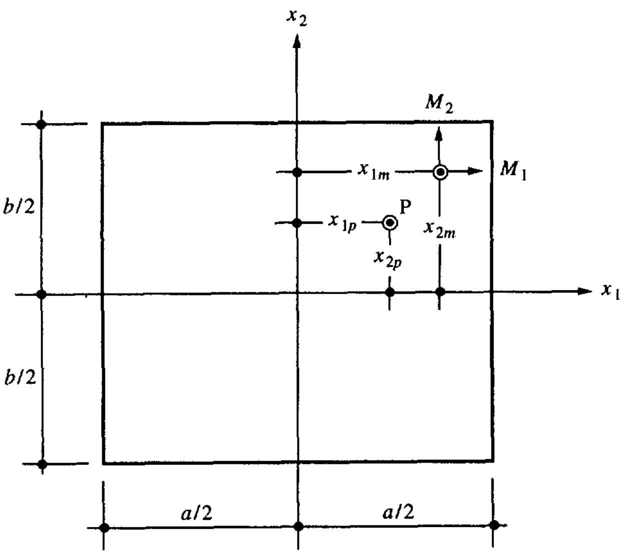

Fig. 1. The form of considered plate and external forces and moments.

are perpendicular to the ox_1 and ox_3 axes and u_1 and u_2 are the components of the displacement on the plane $x_3 = \text{const.}$ in the direction ox_1 and ox_2, respectively.

Taking into account (1), the linear geometrical and constitutive relations, the equilibrium equation of the considered plate can be written as follows:

$$L_{j1}W + L_{j2}\Psi_1 + L_{j3}\Psi_2 = \Big[q(x_1,x_2) + H(W)P_f(x_1,x_2) - P\delta(x_1 - x_{1p})\delta(x_2 - x_{2p})$$

$$+ M_1\delta(x_1 - x_{1m})\frac{\partial}{\partial x_2}\delta(x_2 - x_{2m}) - M_2\delta(x_2 - x_{2m})\frac{\partial}{\partial x_1}\delta(x_1 - x_{1m})\Big]\delta_j^1; \quad j = 1,2,3 \quad (2)$$

where

$$H(W) = 1 \quad \text{for } x_1, x_2 \in \Omega$$

$$H(W) = 0 \quad \text{for } x_1, x_2 \notin \Omega$$

$$L_{11} = 0; \quad L_{12} = D_{11}\frac{\partial}{\partial x_1}; \quad L_{13} = D_{22}\frac{\partial}{\partial x_2}$$

$$L_{21} = D_{11}\frac{\partial^3}{\partial x_1^3} + (D_{12} + 2D_{66})\frac{\partial^3}{\partial x_1 \partial x_2^2}$$

$$L_{22} = D_{11} - h^2\gamma_1\left(D_{11}\frac{\partial^2}{\partial x_1^2} + D_{66}\frac{\partial^2}{\partial x_2^2}\right)$$

$$L_{23} = -h^2\gamma_1\frac{E_2}{10E_1}(D_{12} + D_{66})\frac{\partial^2}{\partial x_1 \partial x_2}$$

$$L_{31} = D_{22}\frac{\partial^3}{\partial x_2^3} + (D_{12} + 2D_{66})\frac{\partial^3}{\partial x_1^2 \partial x_2}$$

$$L_{32} = -h^2\gamma_2\frac{E_2}{10E_1}(D_{12} + D_{66})\frac{\partial^2}{\partial x_1 \partial x_2}$$

$$L_{33} = D_{22} - h^2\gamma_2\left(D_{22}\frac{\partial^2}{\partial x_2^2} + D_{66}\frac{\partial^2}{\partial x_1^2}\right). \quad (3)$$

In (2), (3) the following notation is used: $q(x_1, x_2)$, P are distributed and single loads respectively and P is located at point (x_{1p}, x_{2p}), M_1 and M_2 are the components of the located moment at the point (x_{1m}, x_{2m}), $\delta(x)$ is generalized function, Ω is the unknown contact region, $P_f(x_1, x_2)$ is unknown support reaction of the foundation, δ_j^1 are Kronecker symbols. In (2), (3) are also used the following notations:

$$\gamma_i = \frac{E_1}{G_{i3}(1 - \mu_{12}\mu_{21})}, \quad i = 1,2$$

$$\varphi_i(x_1, x_2) = \gamma_i\Psi_i(x_1, x_2)$$

$$D_{11} = \frac{E_1 h^3}{12(1 - \mu_{12}\mu_{21})}, \quad D_{22} = \frac{E_2 h^3}{12(1 - \mu_{12}\mu_{21})}, \quad D_{66} = \frac{G_{12}h^3}{12}, \quad D_{12} = \frac{E_2\mu_{21}}{1 - \mu_{21}\mu_{12}} \quad (4)$$

where E_1 and E_2 are the modulus of elasticities in the directions of ox_1 and ox_2, G_{12}, G_{13} and G_{23} are the shear modulus of elasticities in the plane ox_1x_2, ox_1x_3 and ox_2x_3, respectively, $\mu_{ij}(i,j = 1,2)$ is the Poisson's ratios. It is necessary to add to the eqns (2), (3) the free edge conditions for considered plate which can be written as follows according to Timoshenko and Woinowsky-Krieger (1959):

$$M_{11} = 0,\quad V_1 = Q_1 - \frac{\partial M_{12}}{\partial x_2} = 0 \quad \text{at}\, x_1 = \pm a/2$$

$$M_{22} = 0,\quad V_2 = Q_2 - \frac{\partial M_{12}}{\partial x_1} = 0 \quad \text{at}\, x_2 = \pm b/2$$

$$R = 0 \quad \text{at}\ x_1 = \pm a/2,\quad x_2 = \pm b/2. \tag{5}$$

In (5) M_{11} and M_{22} are the bending moments, Q_1 and Q_2 are the normal shear forces, M_{12} is the twisting moment, R is the corner force. We write the representation of the M_{11}, M_{22}, V_1, V_2 and R by the W, Ψ_1 and Ψ_2 as follows:

$$M_{11} = -D_{11}\left[\frac{\partial^2 W}{\partial x_1^2} + \mu_{12}\frac{\partial^2 W}{\partial x_2^2} - h^2\gamma_1\left(\frac{\partial \Psi_1}{\partial x_1} + \mu_{12}\frac{\gamma_2}{\gamma_1}\frac{\partial \Psi_2}{\partial x_2}\right)\right]$$

$$M_{22} = -D_{22}\left[\frac{\partial^2 W}{\partial x_2^2} + \mu_{21}\frac{\partial^2 W}{\partial x_1^2} - h^2\gamma_2\left(\frac{\partial \Psi_2}{\partial x_2} + \mu_{21}\frac{\gamma_1}{\gamma_2}\frac{\partial \Psi_1}{\partial x_1}\right)\right]$$

$$V_1 = -D_{11}\left[\frac{\partial^3 W}{\partial x_1^3} + \left(4\frac{D_{66}}{D_{11}} + \mu_{12}\right)\frac{\partial^3 W}{\partial x_1 \partial x_2^2} - h^2\gamma_1\left(\frac{\partial^2 \Psi_1}{\partial x_1^2} + \frac{2D_{66}}{D_{11}}\frac{\partial^2 \Psi_1}{\partial x_2^2}\right)\right]$$

$$-\gamma_2 h^2\left(\mu_{12} + \frac{2D_{66}}{D_{11}}\right)\frac{\partial^2 \Psi_2}{\partial x_1 \partial x_2}$$

$$V_2 = -D_{22}\left[\frac{\partial^3 W}{\partial x_2^3} + \left(4\frac{D_{66}}{D_{11}} + \mu_{21}\right)\frac{\partial^3 W}{\partial x_1^2 \partial x_2} - h^2\gamma_2\left(\frac{\partial^2 \Psi_2}{\partial x_2^2} + \frac{2D_{66}}{D_{11}}\frac{\partial^2 \Psi_2}{\partial x_1^2}\right)\right]$$

$$-\gamma_1 h^2\left(\mu_{21} + \frac{2D_{66}}{D_{11}}\right)\frac{\partial^2 \Psi_1}{\partial x_1 \partial x_2}$$

$$R = -2D_{66}\left[2\frac{\partial^2 w}{\partial x_1 \partial x_2} - h^2\gamma_1\left(\frac{\partial \Psi_1}{\partial x_2} + \frac{\gamma_2}{\gamma_1}\frac{\partial \Psi_2}{\partial x_1}\right)\right]. \tag{6}$$

Consider the determination of the surface displacement of elastic half-space under action of the distributed load $P_f(x_1, x_2)$. Note that the representation for this displacement can be written by using the well known Boussinesq solution for the vertical point load in the following form:

$$W_f = \frac{1-\mu_f^2}{E_f}\iint_\Omega \frac{P_f(\xi_1,\xi_2)\mathrm{d}\Omega}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}} \tag{7}$$

where W_f is the surface displacement of the considered half-space, E_f and μ_f are the modulus of elasticity and Poisson's ratio for the foundation, respectively, ξ_1 and ξ_2 are the coordinates of the points of the region Ω.

Moreover, the unknown reaction force $P_f(x_1, x_2)$ must satisfy the following equilibrium condition

$$\int_{-b/2}^{+b/2}\int_{-a/2}^{+a/2}\left[q(x_1,x_2) + P\delta(x_1 - x_{1p})\delta(x_2 - x_{2p}) + M_{11}\delta(x_1 - x_{1m})\frac{\partial}{\partial x_2}\delta(x_2 - x_{2m})\right.$$

$$\left. + M_{22}\delta(x_2 - x_{2m})\frac{\partial}{\partial x_1}\delta(x_1 - x_{1m})\right]\mathrm{d}x_1\mathrm{d}x_2 = \iint_\Omega P_f(x_1,x_2)\mathrm{d}\Omega. \tag{8}$$

Taking into account above-stated we can write the contact condition between the considered plate and elastic half-space in the following form

$$W_f(x_1,x_2) = W(x_1,x_2), \quad P_f(x_1,x_2) > 0 \quad \text{for } x_1,x_2 \in \Omega$$
$$W_f(x_1,x_2) > W(x_1,x_2), \quad P_f(x_1,x_2) = 0 \quad \text{for } x_1,x_2 \notin \Omega. \tag{9}$$

Thus, the investigation of the considered problem is reduced to the solution of the system of eqns (2), (3), (7) with the boundary conditions (5) and the contact conditions (9). If we change the relation (7) with the following one

$$W_f(x_1,x_2) = \frac{1}{K_f} P_f(x_1,x_2) \tag{10}$$

and the relation (9) with

$$W_f(x_1,x_2) = W(x_1,x_2) \quad \text{for } x_1,x_2 \in \Omega$$
$$W_f(x_1,x_2) = 0 \quad \text{for } x_1,x_2 \notin \Omega \tag{11}$$

then, we obtain the formulation of the problem considered for the Winkler type tensionless foundation. Note that in (10) K_f is the stiffness of the Winkler foundation.

With the above-stated, the formulation of the problem considered which is nonlinear is exhausted.

3. THE METHOD OF SOLUTION OF FORMULATED PROBLEM

As it is impossible to obtain the analytical solution of the problem formulated in Section 2, we attempt to look for an approach analytic-numerical solution by applying Galerkin method. Introducing the dimensionless coordinates as

$$\eta_1 = \frac{x_1}{a}, \quad \eta_2 = \frac{x_2}{b} \tag{12}$$

represent the unknown functions W, Ψ_1, Ψ_2 in the following form

$$W = a \sum_{n=1}^{N} \sum_{m=1}^{N} A_{nm} X_{1n}(\eta_1) X_{2m}(\eta_2)$$
$$\Psi_1 = \frac{1}{a^2} \sum_{n=1}^{N} \sum_{m=1}^{N} B_{nm} X'_{1n}(\eta_1) X_{2m}(\eta_2)$$
$$\Psi_2 = \frac{1}{a^2} \sum_{n=1}^{N} \sum_{m=1}^{N} C_{nm} X_{1n}(\eta_1) X'_{2m}(\eta_2). \tag{13}$$

In (13) A_{nm}, B_{nm}, C_{nm} are unknown constants, $X_{in}(\eta_1)$ are the eigenfunctions of unconstrained free beams parallel to the edges. These functions are given by

$$X_{i1}(\eta_i) = 1, \quad X_{i2}(\eta_i) = 2\sqrt{3}\eta_i$$
$$X_{im}(\eta_i) = \sqrt{2/(\cosh^2(\beta_m/2)+\cos^2(\beta_m/2))}\{\cosh(\beta_m/2)\cos(\beta_m\eta_i) + \cos(\beta_m/2)\cosh(\beta_m\eta_i)\} \quad i = 1,2; \quad m = 3,5,7,\ldots,$$
$$X_{im}(\eta_i) = \sqrt{2/(\sinh^2(\beta_m/2)-\sin^2(\beta_m/2))}\{\sinh(\beta_m/2)\sin(\beta_m\eta_i) + \sin(\beta_m/2)\sinh(\beta_m\eta_i)\} \quad i = 1,2; \quad m = 4,6,8,\ldots, \tag{14}$$

where the parameters β_m are obtained by calculating numerically the positive roots of the equation

$$\tan(\beta/2) \pm \tanh(\beta/2) = 0 \quad m = \begin{Bmatrix} 3,5,7,\ldots \\ 4,6,8,\ldots \end{Bmatrix}. \tag{15}$$

Taking into account that,

$$X''_{im}(\eta_i) I_{\eta_i = \pm 1/2} = X'''_{im}(\eta_i) I_{\eta_i = \pm 1/2} = 0 \tag{16}$$

we obtain

$$\begin{aligned}
\left(\frac{\partial^2 W}{\partial x_1^2} - h^2 \gamma_1 \frac{\partial \Psi_1}{\partial x_1}\right)\Bigg|_{x_1 = \pm a/2} &= 0, \\
\left(\frac{\partial^3 W}{\partial x_1^3} - 0.1 h^2 \gamma_1 \frac{\partial^2 \Psi_1}{\partial x_1^2}\right)\Bigg|_{x_1 = \pm a/2} &= 0, \\
\left(\frac{\partial^2 W}{\partial x_2^2} - h^2 \gamma_1 \frac{\partial \Psi_2}{\partial x_2}\right)\Bigg|_{x_2 = \pm b/2} &= 0, \\
\left(\frac{\partial^3 W}{\partial x_2^3} - 0.1 h^2 \gamma_1 \frac{\partial^2 \Psi_2}{\partial x_2^2}\right)\Bigg|_{x_2 = \pm b/2} &= 0.
\end{aligned} \tag{17}$$

Because the boundary conditions (5) are not satisfied automatically under representation of the unknown functions W, Ψ_1, Ψ_2 by the formulae (13), (14), we change the boundary conditions (5) with (17) and add to the first equation of (2) the following "residual forces":

$$R^r = L^r_{11} W + L^r_{12} \Psi_1 + L^r_{13} \Psi_2 \tag{18}$$

where

$$\begin{aligned}
L^r_{11} = {} & D_{11} \mu_{12} \frac{\partial}{\partial x_1} \left\{ [\delta(x_1 + a/2) - \delta(x_1 - a/2)] \frac{\partial^2}{\partial x_2^2} \right\} \\
& + D_{11} \left(4 \frac{D_{66}}{D_{11}} + \mu_{12} \right) \{\delta(x_1 + a/2) - \delta(x_1 - a/2)\} \frac{\partial^3}{\partial x_1 \partial x_2^2} \\
& + D_{22} \mu_{21} \frac{\partial}{\partial x_2} \left\{ [\delta(x_2 + b/2) - \delta(x_2 - b/2)] \frac{\partial^2}{\partial x_1^2} \right\} \\
& + D_{22} \left(4 \frac{D_{66}}{D_{11}} + \mu_{21} \right) \{\delta(x_2 + b/2) - \delta(x_2 - b/2)\} \frac{\partial^3}{\partial x_1^2 \partial x_2};
\end{aligned}$$

$$\begin{aligned}
L^r_{12} = {} & -D_{66} \frac{\gamma_1}{5} \frac{\partial^2 \Psi_1}{\partial x_2^2} \{\delta(x_1 + a/2) - \delta(x_1 - a/2)\} \\
& - D_{22} \frac{h^2}{10} \gamma_1 \mu_{21} \frac{\partial}{\partial x_2} \left\{ \frac{\partial \Psi_1}{\partial x_1} [\delta(x_2 + b/2) - \delta(x_2 - b/2)] \right\} \\
& - D_{66} \frac{h^2 \gamma_1}{5} \frac{\delta \Psi_1}{\partial x_2} \{\delta(x_1 + a/2) - \delta(x_1 - a/2)\} \{\delta(x_2 + b/2) - \delta(x_2 - b/2)\};
\end{aligned}$$

$$L'_{13} = -D_{11}\frac{h^2}{10}\gamma_2\mu_{12}\frac{\partial}{\partial x_1}\left\{\frac{\partial \Psi_2}{\partial x_2}\delta(x_1+a/2)-\delta(x_1-a/2)\right\}$$

$$+D_{22}\frac{D_{66}}{D_{11}}\gamma_2\frac{h^2}{5}\frac{\partial^2\Psi_1}{\partial x_1^2}\{\delta(x_2+b/2)-\delta(x_2-b/2)\}$$

$$-D_{66}\gamma_2\frac{\partial\Psi_2}{\partial x_1}\{\delta(x_1+a/2)-\delta(x_1-a/2)\}\{\delta(x_2+b/2)-\delta(x_2-b/2)\}. \quad (19)$$

Thus, the investigation of the problem formulated in Section 2 is reduced to the solution of the eqns (2), (3), (7) (with taking into account the changing L_{1i} to $L_{1i}+L'_{1i}$) under boundary conditions (17) and contact condition (9). Further, we take the first order derivative of the second and third equation of (2) with respect to x_1, x_2, respectively. Substituting the representations (13) into these equations and carrying out the well known procedures, we obtain the following system of equations with respect to the unknown constants entering to (13) and function P_f:

$$\sum_{n=1}^{N}\sum_{m=1}^{N} B_{nm}b_{nmkl}+\sum_{m=1}^{N} C_{nm}c_{nmkl} = f_{kl}\left(\frac{a}{h}\right)^3+\frac{E_f}{E_1}\iint_\Omega \frac{P_f}{E_f}X_k(\eta_1)X_l(\eta_2)\mathrm{d}\Omega$$

$$\sum_{n=1}^{N}\sum_{m=1}^{N}(A_{nm}a_{1nmkl}+B_{nm}b_{1nmkl}+C_{nm}c_{1nmkl}) = 0$$

$$\sum_{n=1}^{N}\sum_{m=1}^{N}(A_{nm}a_{2nmkl}+B_{nm}b_{2nmkl}+C_{nm}c_{2nmkl}) = 0 \quad (20)$$

$$\frac{1-\mu_f^2}{\pi}\iint_\Omega \frac{P_f(\xi_1,\xi_2)\mathrm{d}\Omega}{E_f\sqrt{(\eta_1-\xi_1)^2+(\eta_2-\xi_2)^2}} = \sum_{n=1}^{N}\sum_{n=1}^{N} A_{nm}X_{1n}(\eta_1)X_{2m}(\eta_2), \quad k,l=1,2,\dots N \quad (21)$$

where

$$b_{nmkl} = d_{11}\left(\frac{h}{a}\right)^3\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} X''_{1n}(\eta_1)X_{2m}(\eta_2)X_{1k}(\eta_1)X_{2l}(\eta_2)\mathrm{d}\eta_1\mathrm{d}\eta_2,$$

$$c_{nmkl} = d_{22}\beta\left(\frac{h}{a}\right)^3\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} X_{1n}(\eta_1)X''_{2m}(\eta_2)X_{1k}(\eta_1)X_{2l}(\eta_2)\mathrm{d}\eta_1\mathrm{d}\eta_2,$$

$$a_{1nmkl} = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}[d_{11}X^{iv}_{1n}(\eta_1)X_{2m}(\eta_2)+(d_{12}+2d_{66})\beta^2X''_{1n}(\eta_1)X''_{2m}(\eta_2)]$$
$$\times X_{1k}(\eta_1)X_{2l}(\eta_2)\mathrm{d}\eta_1\mathrm{d}\eta_2$$

$$b_{1nmkl} = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\left[d_{11}X''_{1n}(\eta_1)X_{2m}(\eta_2)-\left(\frac{h}{a}\right)^2\gamma_1 d_{11}X'''_{1n}(\eta_1)X_{2m}(\eta_2)\right.$$
$$\left.+d_{66}\beta^2X'_{1n}(\eta_1)X''_{2m}(\eta_2)\right]X_{1k}(\eta_1)X_{2l}(\eta_2)\mathrm{d}\eta_1\mathrm{d}\eta_2$$

$$C_{1nmkl} = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\frac{E_2}{E_1}(d_{12}+d_{66})\beta X''_{1n}(\eta_1)X''_{2m}\mathrm{d}\eta_1\mathrm{d}\eta_2\,;$$

$$a_{2nmkl} = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} [d_{22}\beta^3 X_{1n}(\eta_1)X_{2m}^{iv}(\eta_2)X_{1k}(\eta_1)X_{2l}(\eta_2)$$

$$+(d_{12}+2d_{66})\beta A_{nm}X''_{1n}(\eta_1)X''_{2m}(\eta_2)X_{1k}(\eta_1)X_{2l}(\eta_2)]\mathrm{d}\eta_1\mathrm{d}\eta_2\,;$$

$$b_{2nmkl} = \frac{E_2}{E_1}(d_{12}+d_{66})\beta\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} X''_{1n}(\eta_1)X''_{2m}(\eta_2)X_{1k}(\eta_1)X_{2l}(\eta_2)\mathrm{d}\eta_1\mathrm{d}\eta_2\,;$$

$$C_{2nmkl} = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\left[d_{22}X_{1n}(\eta_1)X''_{2m}(\eta_2) - \left(\frac{h}{a}\right)^2 \gamma_2 d_{22}\beta^2 X_{1n}(\eta_1)X'''_{2m}(\eta_2)\right.$$

$$\left.+d_{66}X''_{1n}(\eta_1)X''_{2m}(\eta_2)\right]X_{1k}(\eta_1)X_{2l}(\eta_2)\mathrm{d}\eta_1\mathrm{d}\eta_2\,;$$

$$f_{kl} = \frac{1}{E_1}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} F(\eta_1,\eta_2)X_l(\eta_1)X_k(\eta_2)\mathrm{d}\eta_1\mathrm{d}\eta_2$$

$$F(\eta_1,\eta_2) = q(\eta_1,\eta_2)+P\delta(\eta_1-\eta_{1p})\delta(\eta_2-\eta_{2p})+M_x\delta(\eta_1-\eta_{1m})\frac{\delta}{\delta\eta_2}\delta(\eta_2-\eta_{2m})$$

$$+M_y\delta(\eta_2-\eta_{2m})\frac{\partial}{\partial\eta_1}\delta(\eta_1-\eta_{1m}). \tag{22}$$

For Winkler type foundation, using (10), (11) we change the last equation of (21) with the following one:

$$P_f(\eta_1,\eta_2) = K_f\sum_{n=1}^{N}\sum_{m=1}^{N} A_{nm}X_{1n}(\eta_1)X_{2m}(\eta_2) \quad \text{for } a\eta_1, b\eta_2 \in \Omega$$

$$P_f(\eta_1,\eta_2) = 0 \quad \text{for } a\eta_1, b\eta_2 \notin \Omega. \tag{23}$$

Thus, the solution of the considered nonlinear contact problems for the system of anisotropic (orthotropic) plates and elastic half-space (and Winkler foundation) are reduced to the solution of the eqns (20)–(22), ((20), (23) and (22)).

4. NUMERICAL SOLUTION PROCEDURE

As the considered system of eqns (20)–(22) or (20), (23), (22) are nonlinear, the numerical solution of this system are carried out by using an iteration procedure. In this case, in the first iteration, we suppose that there exist full contact between plate and foundation and define such part of the contact region under which $W(x_1,x_2) < 0$ (for the Winkler foundation) or $P_f(x_1,x_2) < 0$ (for the elastic half-space). The equation $W(x_1,x_2) = 0$ for the Winkler foundation and the equation $P_f(x_1,x_2) = 0$ for the elastic half-space is defined the boundary of the contact region $\Omega_1 \subset \{x_1,x_2;\ x_1\in[-a/2,a/2],\ x_2\in[-b/2,b/2]\}$. In the second iteration we suppose that the full contact between plate and foundation is fulfilled only on the Ω_1 and for x_1, $x_2\in\Omega_1\setminus\{x_1,x_2;\ x_1\in[-a/2,a/2];\ x_2\in[-b/2,b/2]\}$, $P_f(x_1,x_2) = 0$. Reasoning as in the first iteration, we define the refined contact region $\Omega_2 \subset \Omega_1$. The above described iterations is continued up to $W(x_1,x_2) > 0$ (for the Winkler foundation) or $P_f(x_1,x_2) > 0$ (for the elastic half-space) for all x_1, $x_2\in\Omega_k$, where k is the number of the last iteration. Thus, we determine the contact region as $\Omega = \Omega_k$ between the anisotropic plate and the "tensionless" foundation. After determination of the contact region Ω, the others seeking values are also determined.

Note that in the case when the anisotropic plate rests on the elastic half-space under the procedure of carrying out of the above-stated iterations, it is necessary to algebrize the singular integral eqn (21) for the numerical solution. For this purpose, we act in the following form.

It is known that, if the edges and the corners of the plate are in contact with the elastic half-space, the contact pressure is singular in these places. When the plate considered is a rigid one, the most singularities arise. In the last case, the order of the singularities at the edges and the corners are equal to 0.5 and α, respectively. Note that, the value of α is calculated by many authors and the different but near values has been obtained for it. For example, in the papers of Rvachev (1959), Bazant (1974) and Borodachev (1976) it has been obtained as $\alpha = 0.686$, $\alpha = 0.704$ and $\alpha = 0.6996$ respectively. In our numerical investigations $\alpha = 0.7$ is assumed and the unknown function $P_f(x_1, x_2)$ is represented in the following form:

$$P_f(x_1, x_2) = \bar{P}_f(x_1, x_2)\frac{a^{2\alpha}[(x_1+a/2)(a/2-x_1)+(x_2+b/2)(b/2-x_2)]^{1-\alpha}}{[(x_1+a/2)(a/2-x_1)(x_2+b/2)(b/2-x_2)]^{1/2}} \tag{24}$$

where $\bar{P}_f(x_1, x_2)$ is bounded function at the contact region.

Further, the contact region Ω is divided into small elements Q_j ($j = 1, \ldots$ N$_1$) by assuming that $\bar{P}_f(x_1, x_2)$ is constant in each element Q_j. So, we may write the following relation:

$$I(\eta_1, \eta_2) = \iint_{\Omega} P_f(\xi_1, \xi_2) g(\eta_1, \eta_2, \xi_1, \xi_2)\mathrm{d}\xi_1 \mathrm{d}\xi_2$$

$$= \sum_{j=1}^{N} \bar{P}_f(\xi_{1j}, \xi_{2j}) \iint_{\Omega_j} \frac{\beta\left[(\xi_1+1/2)(1/2-\xi_1)+\frac{1}{\beta^2}(\xi_2+1/2)(1/2-\xi_2)\right]^{1-\alpha}}{[(\eta_1+1/2)(1/2-\eta_1)(\eta_2+1/2)(1/2-\eta_2)]^{1/2}} \times g(\eta_1, \eta_2, \xi_1, \xi_2)\mathrm{d}\xi_1 \mathrm{d}\xi_2. \tag{25}$$

Note that, under writing the relation (25), the dimensionless coordinates (12) is used and the notation $\beta = a/b$ is introduced.

Assuming the compatibility conditions (i.e., the eqn (21) hold at the center of each element Ω_j) we obtain the following equations from (21), (25):

$$I(\eta_{1j}, \eta_{2j}) = \sum_{n=1}^{N_1}\sum_{m=1}^{N_1} A_{nm} X_{1n}(\eta_{1j}) X_{2m}(\eta_{2j}) \quad j = 1, 2, \ldots, N_1, \eta_{1j}, \eta_{2j} \in Q_j. \tag{26}$$

Thus, the singular integral eqn (21) is algebrized as (26).

Consider the choice of the value of N which enter to the eqns (13), (14), (21), (23) and of the value N_1 which enter to the eqn (26). Not that, the choice of the N and N_1 in our investigations is based on the numerical convergence of the obtained numerical results in each integration. In this case we require the difference between the numerical results obtained under N and N_1 and those obtained under $N-1$ and N_1-1 must not be more than 0.05%.

5. NUMERICAL RESULTS AND DISCUSSIONS

Now, consider some numerical results which are obtained by using the above-described algorithm in the case, when only the external force P acts at the point $x_1 = x_2 = 0$. Note that the numerical results which will be considered are obtained in the framework five iteration and the values of N and N_1 are defined in each iteration by using above-stated requirement for N and N_1.

Consider a square plate, i.e., consider the case when $a = b$ (Fig. 1). Introducing the parameter

$$\Lambda = \left(\frac{h}{a}\right)^2 \frac{E_1}{G_{13}(1-\mu_{12}\mu_{21})}$$

and assume that $\mu_f = 0.1$; $G_{13} = G_{23}$, $G_{12}/E_1 = 0.4$, $\mu_{12} = \mu_{21} = 0.25$, $K_f a^4/D_{11} = 16{,}000$; $E_f a^3/D_{11} = 16{,}000$.

Investigate the influence of the change of E_2/E_1 and Λ on the size and form of the contact region between plate and elastic half-space and on the displacement distribution of the plate and foundation surface. Consider the following two cases:

1. The foundation is modelled as the generalized Winkler foundation (W.F.) which reacts in compression only.
2. The foundation is modelled as the elastic half-space (E.H.S.) which reacts to the compression only in the contact region.

Note that, in the considered external loading, the form of the plate secure the symmetry of the obtained results with respect to the ox_1 and ox_2 axes (Fig. 1). So, we will study only on the first quadrature below.

Thus, consider the Fig. 2, on which, at various E_2/E_1 and Λ, the contact region between plate and elastic foundation are shown. Note that, the results showed in Fig. 2 in the case when $E_2/E_1 = 1$, $\Lambda = 0$ for the Winkler foundation coincide with the results of Celep (1988), but for the elastic half-space coincide with the results of Li and Dempsey (1988).

Moreover, the obtained results show that under $E_2/E_1 = 1$ by increasing Λ, the size of the contact region decrease and the form of these regions is a circle. The comparison of the

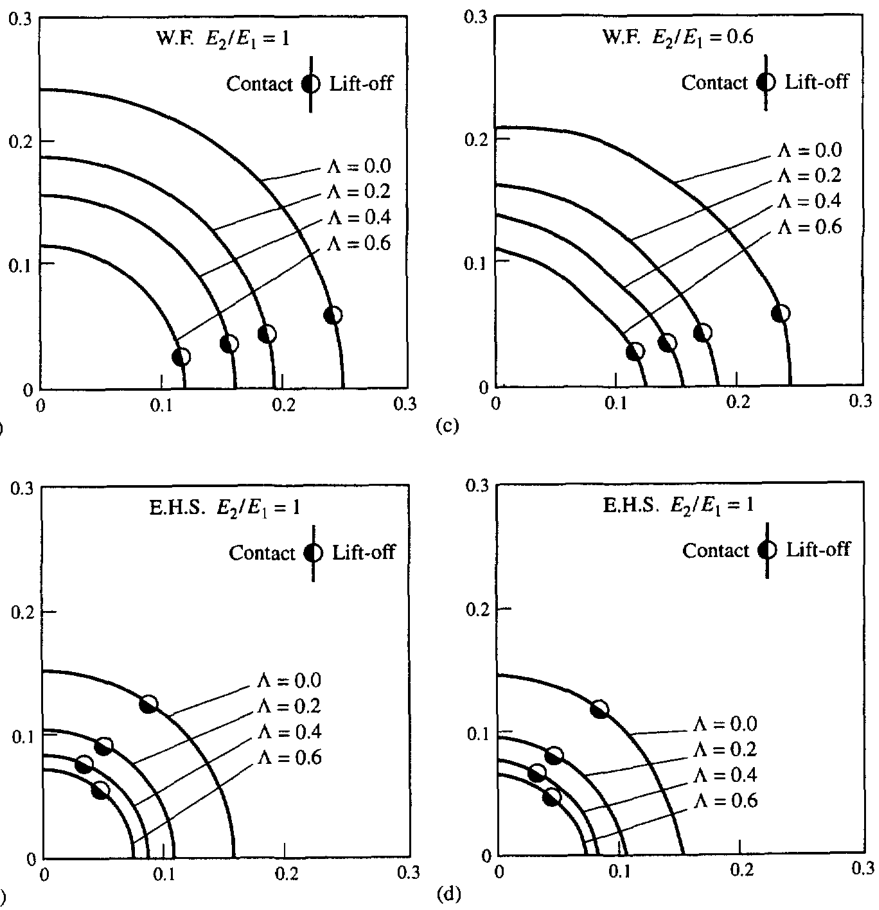

Fig. 2. The influence of considered problem parameters to the size and form of the contact region.

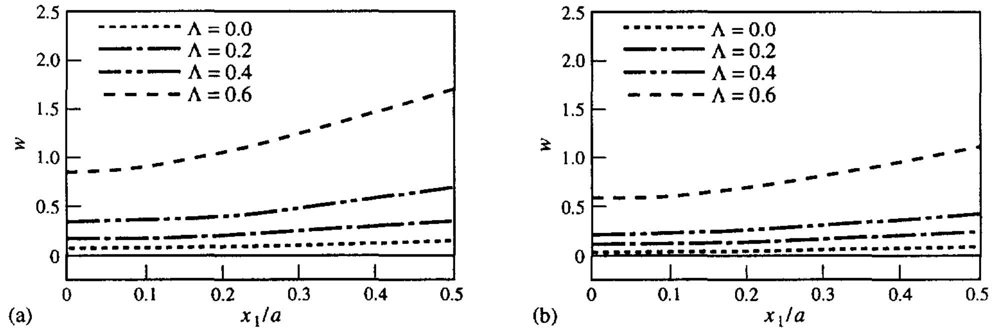

Fig. 3. Displacement distribution with respect to various Λ values for $E_2/E_1 = 1$ along the x_1/a axis at $x_2/b = 1/2\ (-1/2)$, (a) for W.F., (b) for E.H.S.

results obtained for the Winkler foundation with the results obtained for the elastic half-space shows that, in the second case (E.H.S.), the radius of the contact region is smaller from that of which is found in the first case (W.F.).

The above-stated situation occurs also in the case when $E_2/E_1 = 0.6$. But, in this case, the form of the contact region is an ellipse. Moreover, the major axis of this ellipse is in the direction of ox_1 axis, and the minor axis of it is in the ox_2 direction. These results are explained in the case when the modulus of elasticity in the ox_1 direction is bigger than that of in the ox_2 direction, i.e., $E_1 > E_2$.

Introduce the denoting $w = \text{W.P. } 10^2/(a.E_1)$ and consider the influence of the plate material property to the distribution of the displacement at the plate edges. The graphs of the distribution of w with respect to x_1 and x_2 under various Λ are shown in Figs 3–7 with $E_2/E_1 = 1$; 0.9 ; 0.8 ; 0.7 ; 0.6, respectively. Note that the graphs in Figs 3a, 4a, 4b, 5a, 5b, 6a, 6b, 7a, 7b are constructed for the case 1 (for the W.F.) and the graphs in Figs 3b, 4c, 4d, 5c, 5d, 6c, 6d, 7c, 7d are constructed for the case 2 (for the E.H.S.). Moreover, note

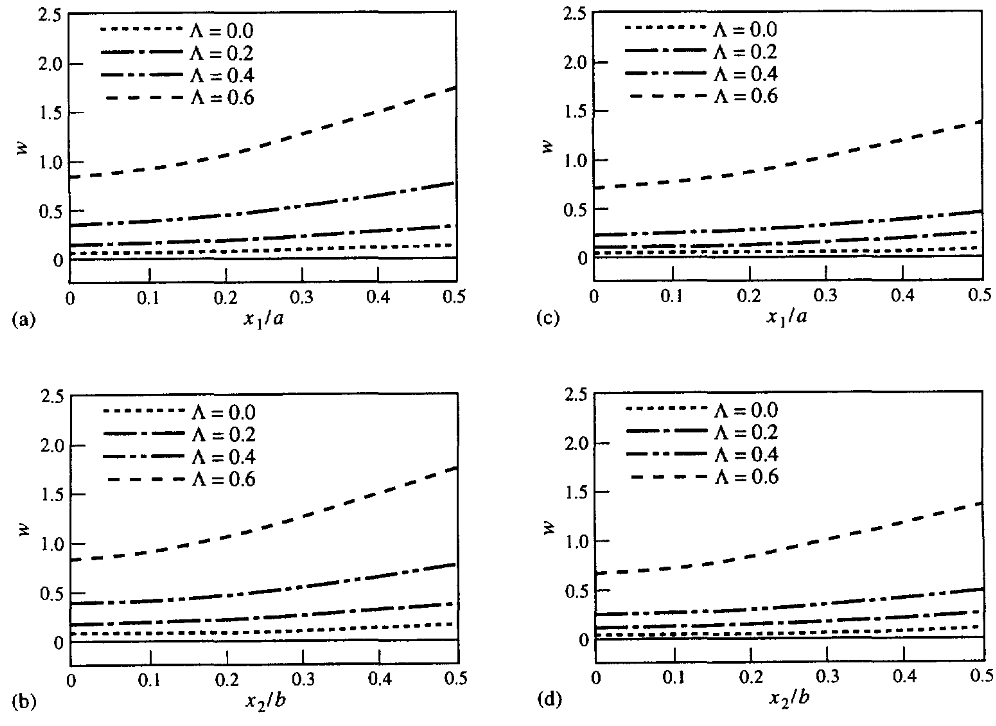

Fig. 4. Displacement distribution with respect to various Λ values for $E_2/E_1 = 0.9$, (a) along the x_1/a axis at $x_2/b = 1/2\ (-1/2)$ for W.F., (b) along the x_2/b axis at $x_1/a = 1/2\ (-1/2)$ for W.F., (c) along the x_1/a axis at $x_2/b = 1/2\ (-1/2)$ for E.H.S., (d) along the x_2/b axis at $x_1/a = 1/2\ (-1/2)$ for E.H.S.

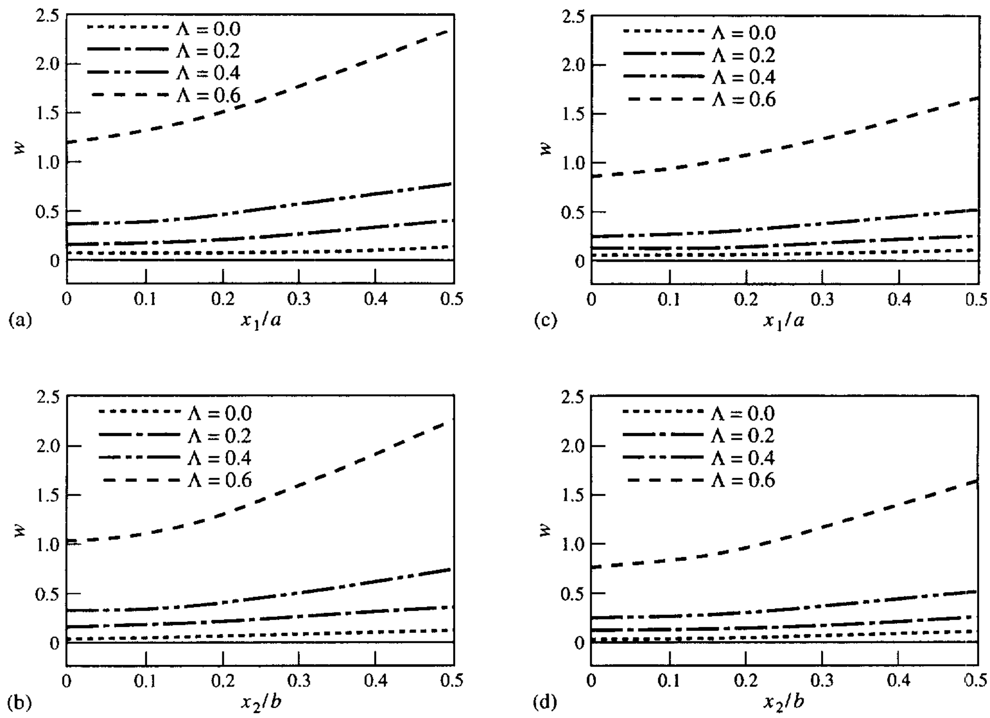

Fig. 5. Displacement distribution with respect to various Λ values for $E_2/E_1 = 0.8$, (a) along the x_1/a axis at $x_2/b = 1/2$ $(-1/2)$ for W.F., (b) along the x_2/b axis at $x_1/a = 1/2$ $(-1/2)$ for W.F., (c) along the x_1/a axis at $x_2/b = 1/2$ $(-1/2)$ for E.H.S., (d) along the x_2/b axis at $x_1/a = 1/2$ $(-1/2)$ for E.H.S.

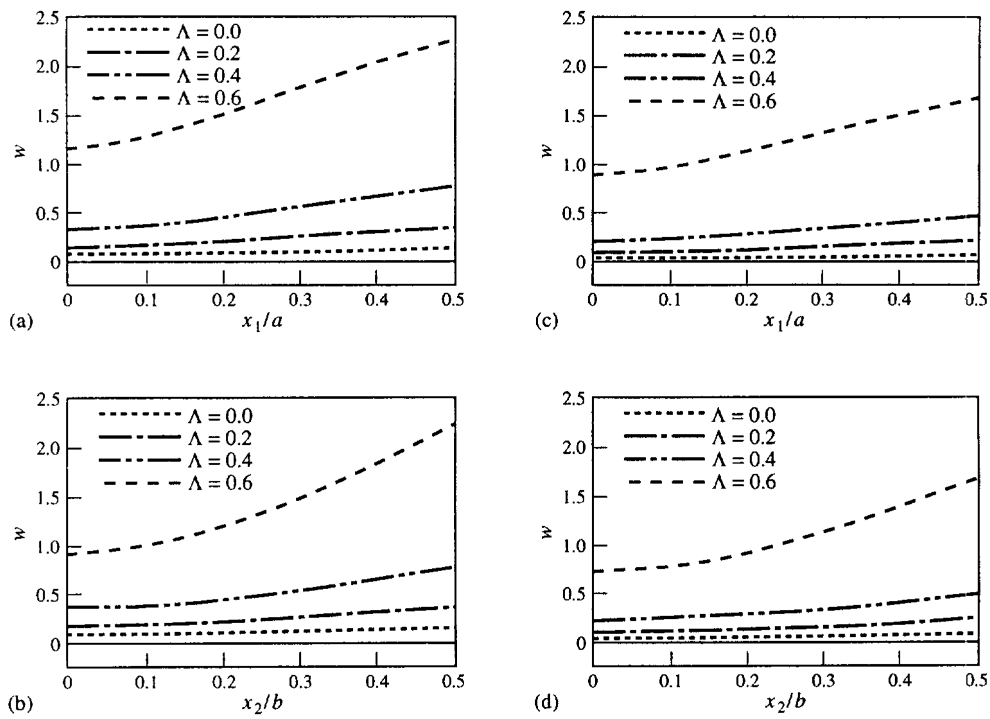

Fig. 6. Displacement distribution with respect to various Λ values for $E_2/E_1 = 0.7$, (a) along the x_1/a axis at $x_2/b = 1/2$ $(-1/2)$ for W.F., (b) along the x_2/b axis at $x_1/a = 1/2$ $(-1/2)$ for W.F., (c) along the x_1/a axis at $x_2/b = 1/2$ $(-1/2)$ for E.H.S., (d) along the x_2/b axis at $x_1/a = 1/2$ $(-1/2)$ for E.H.S.

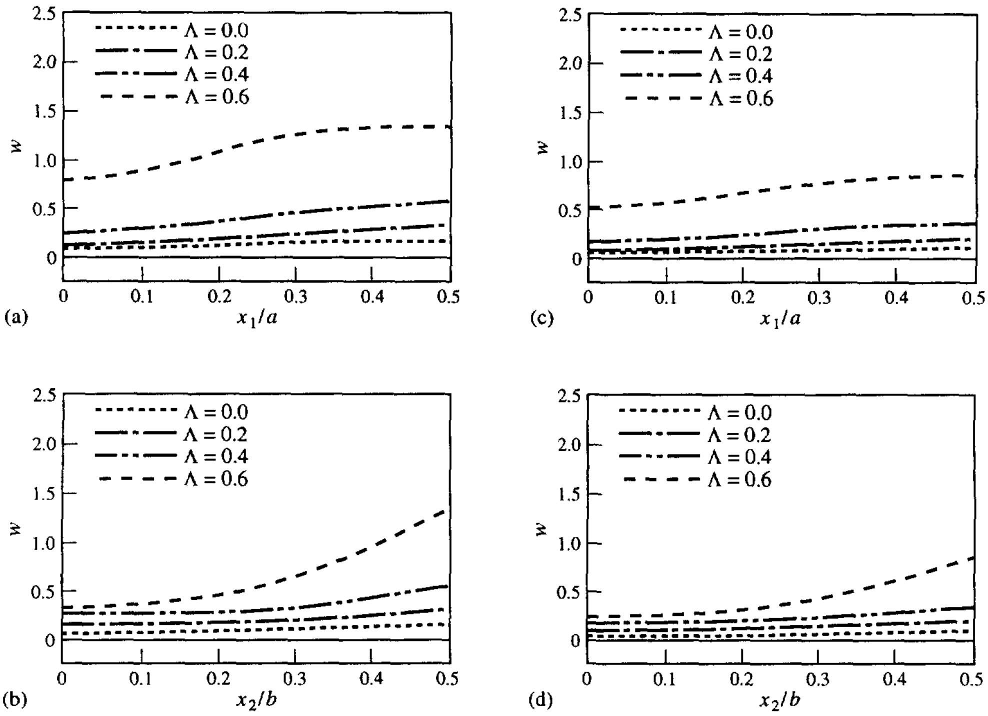

Fig. 7. Displacement distribution with respect to various Λ values for $E_2/E_1 = 0.6$, (a) along the x_1/a axis at $x_2/b = 1/2$ $(-1/2)$ for W.F., (b) along the x_2/b axis at $x_1/a = 1/2$ $(-1/2)$ for W.F., (c) along the x_1/a axis at $x_2/b = 1/2$ $(-1/2)$ for E.H.S., (d) along the x_2/b axis at $x_1/a = 1/2$ $(-1/2)$ for E.H.S.

that the numerical results shown the dependencies between w and x_1/a (and x_2/b) are obtained at $x_2/b = 0.5$ (at $x_1/a = 0.5$).

Analysis of the obtained numerical results shows that with decreasing Λ the values w increase monotonously in all considered cases and the influence of the change of Λ to the values of w is very significant. It follows that the displacement w obtained in the case when the foundation is modelled as W.F. greater than the corresponding displacement w obtained in the case when the foundation is modelled as E.H.S.

The comparison of the numerical results obtained for various E_2/E_1 shows that the influence of the change of E_2/E_1 is non-monotonous. In the considered cases at first with decreasing E_2/E_1 the values w increase. After some values of E_2/E_1 (for the considered cases after $E_2/E_1 = 0.7$) with decreasing E_2/E_1 the values of w decrease. Moreover, with decreasing E_2/E_1 the character of the dependencies between w and x_1/a (at $x_2/b = 0.5$) change, i.e., in the cases $E_2/E_1 = 1$; 0.9; 0.8; 0.7 the maximal value of w are obtained at the plate corner, however in the case $E_2/E_1 = 0.6$ the point at which w is maximal moves from plate corner into the edge $x_2/b = 0.5$.

Now consider the influence of the change of the plate material property to the values of w obtained at $x_1/a = 0$ and at $x_2/b = 0$. The graphs of the dependencies between w and x_1/a at $x_2/b = 0$ and between w and x_2/b at $x_1/a = 0$ are shown in Figs 8–10 with $E_2/E_1 = 1$; 0.8; 0.6, respectively.

Note that the results shown in Figs 8a, 9a, 9b, 10a, 10b correspond to the case 1 (W.F.) and the results shown in the Figs 8b, 9c, 9d, 10c, 10d correspond to the case 2 (E.H.S.).

The analysis of the graphs constructed in Figs 8–10 shows that in all considered cases with decreasing Λ the values w increase monotonously in contact and in lift-off region. Moreover, non-monotonous character of the influence of the change of E_2/E_1 to the values w obtained at $x_1/a = 0$ and at $x_2/b = 0$ is also observed from the above graphs.

Note that, in the case when the foundation is modelled as W.F. the graphs show the displacement distribution of the plate considered with respect to x_1/a at $x_2/b = 0$ or with respect to x_2/b at $x_1/a = 0$ and also show the displacement of foundation surface in the

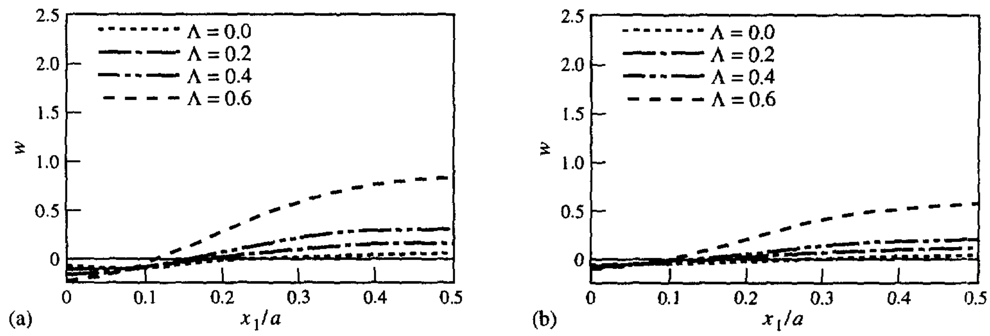

Fig. 8. Displacement distribution with respect to various Λ values for $E_2/E_1 = 1.0$ along the x_1/a axis at $x_2/b = 0$, (a) for W.F., (b) for E.H.S.

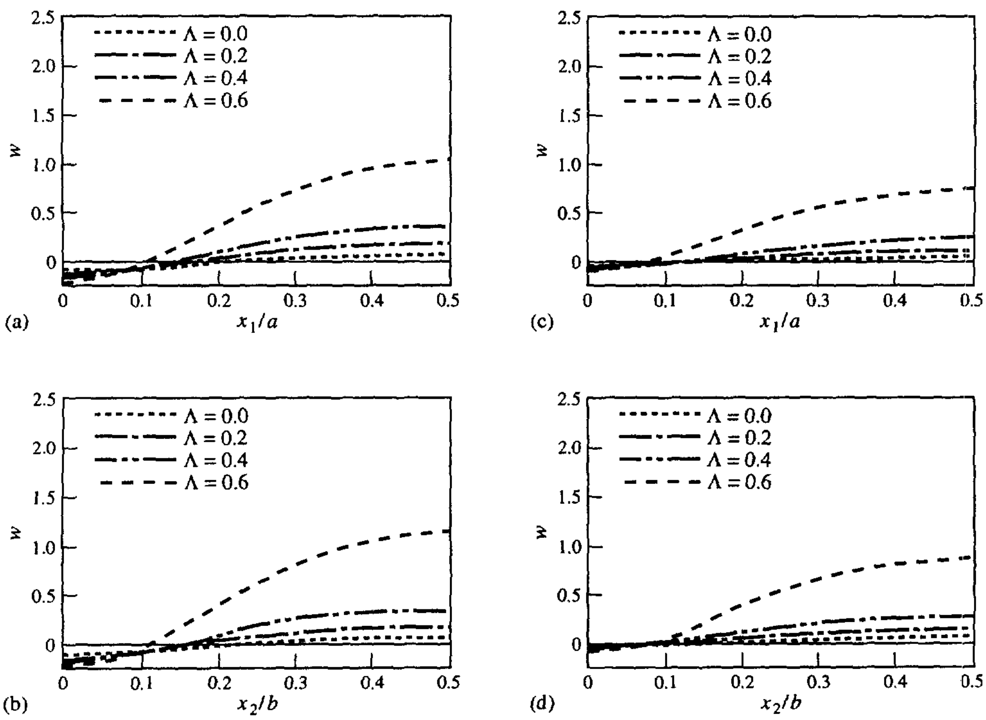

Fig. 9. Displacement distribution with respect to various Λ values for $E_2/E_1 = 0.8$, (a) along the x_1/a axis at $x_2/b = 0$ for W.F., (b) along the x_2/b axis at $x_1/a = 0$ for W.F., (c) along the x_1/a axis at $x_2/b = 0$ for E.H.S., (d) along the x_2/b axis at $x_1/a = 0$ for E.H.S.

contact region (in the lift-off region the displacement of the foundation is equal to zero). However, the graphs corresponded to the case when the foundation is modelled as E.H.S. don't show the displacement of the surface point of E.H.S. For the non-complication of the observation of the plate displacement graphs the numerical results obtained for the surface displacement of the E.H.S. are not indicated in the above figures. These results are given with graphs shown in Fig. 11, which are constructed in the case $E_2/E_1 = 0.8$; $x_1/a = 0$ with various Λ. It follows that with decreasing Λ causes the displacement of the surface point of E.H.S. growth. Note that in the other considered cases analogous results are obtained for the surface displacement of E.H.S., which also follows from the results given in Table 1. The data shown in Table 1 indicate the values w at the centre of the plate considered for various values Λ and E_2/E_1. Moreover, it follows that at the centre of the plate the values of w obtained for the E.H.S. significantly less than those obtained for the W.F. from numerical results given in the Table 1.

The above-shown character of the influence of the change of the parameter Λ (which characterise the shear stiffness of the considered plate material) on the displacement of the

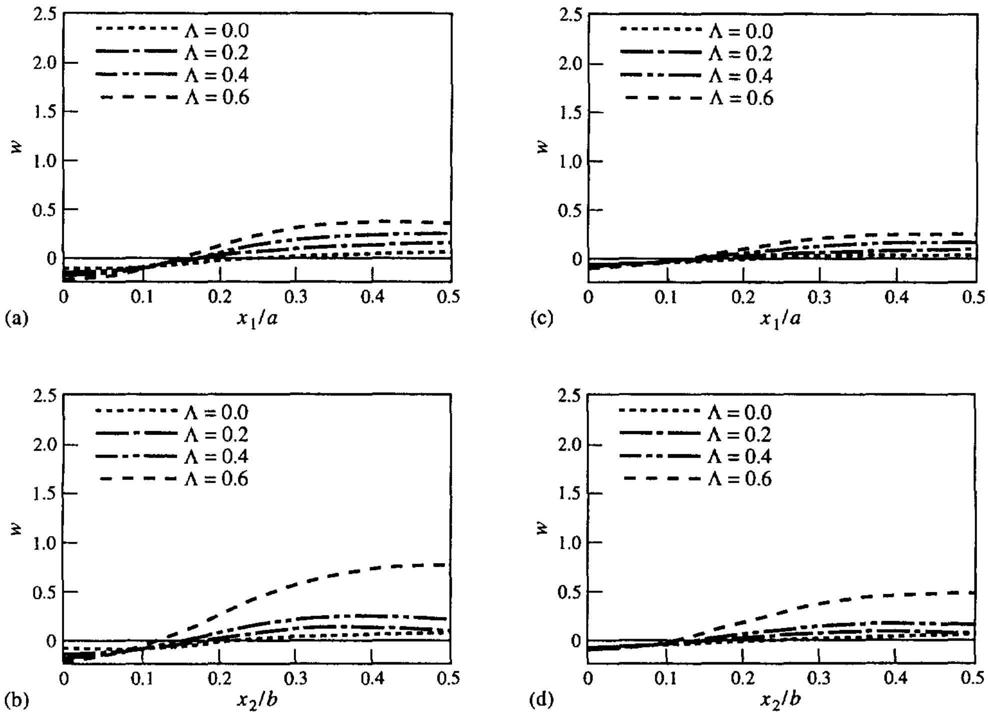

Fig. 10. Displacement distribution with respect to various Λ values for $E_2/E_1 = 0.6$, (a) along the x_1/a axis at $x_2/b = 0$ for W.F., (b) along the x_2/b axis at $x_1/a = 0$ for W.F., (c) along the x_1/a axis at $x_2/b = 0$ for E.H.S., (d) along the x_2/b axis at $x_1/a = 0$ for E.H.S.

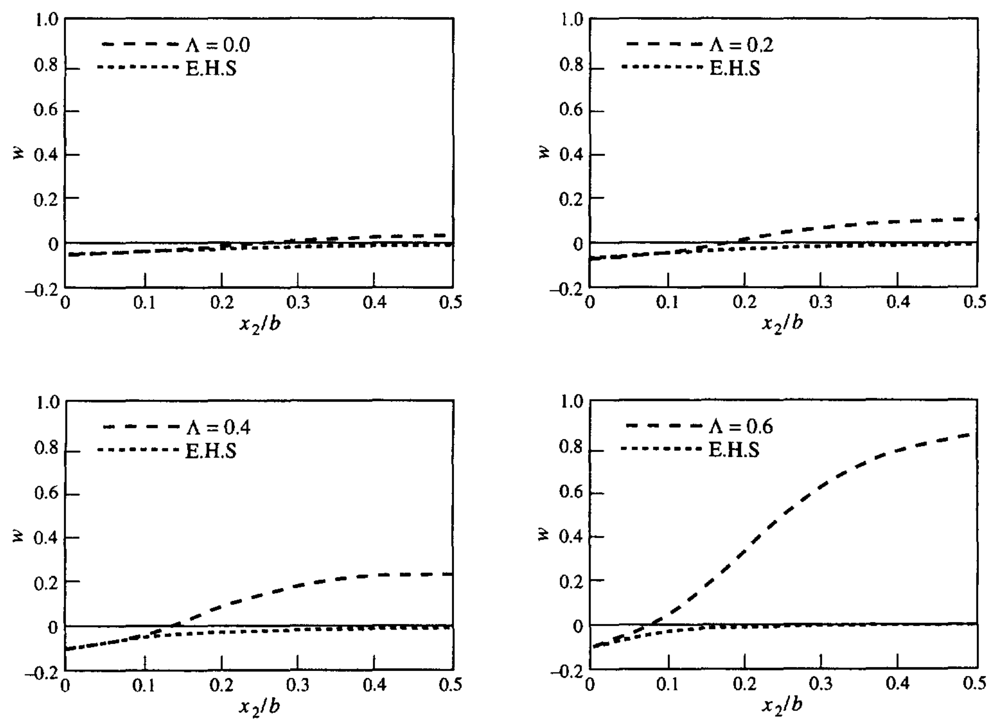

Fig. 11. Displacement distribution of the plate and E.H.S. for $E_2/E_1 = 0.8$ along the x_2/b axis at $x_1/a = 0$ for (a) $\Lambda = 0.0$, (b) $\Lambda = 0.2$, (c) $\Lambda = 0.4$, (d) $\Lambda = 0.6$.

plate considered and on the displacement of the surface of the foundation, agrees with the engineering intuition and with the numerous results obtained for the composite plates under investigations of their various mechanical behaviours. Moreover, by direct verification it is

Table 1. The values of w = W.P. $10^2/(aE_1)$ at center of the plate

Λ	E_2/E_1	W.F.	E.H.S.
0	1.0	0.0784	0.0563
	0.8	0.0813	0.0575
	0.6	0.0854	0.0591
0.2	1.0	0.1262	0.0742
	0.8	0.1325	0.0764
	0.6	0.1358	0.0776
0.4	1.0	0.1664	0.0861
	0.8	0.1754	0.0890
	0.6	0.1724	0.0888
0.6	1.0	0.2100	0.0992
	0.8	0.2340	0.1075
	0.6	0.2177	0.1021

established that the values w obtained in the present paper with $\Lambda = 0$ under $E_2/E_1 = 1$ in case 1 (W.F.) coincide with the corresponding values w obtained in the paper of Celep (1988), in case 2 (E.H.S.) coincide with the corresponding values w obtained in the paper of Li and Dempsey (1988) and under $E_2/E_1 < 1$ in case 1 (W.F.) coincide with those obtained in the paper of Kocatürk (1995).

Again return to the discussion of the influence of the change of E_2/E_1 to the behaviour of the bending of the considered anisotropic plate resting on the foundation which react in compression only. Note that in the present paper under all numerical investigations we suppose that $0.6 \leqslant E_2/E_1 \leqslant 1$ and obtained numerical results show non-monotonous character of the discussed influence. Moreover, the numerical results obtained in the case when $E_2/E_1 < 0.6$ (for example, when $E_2/E_1 = 0.4$) have shown us that in these cases the contact region between plate and tensionless foundation is multi-connected region. Such phenomena for above-described modelling of the foundation under considered loading of the plate are observed only in the case when the plate material is anisotropic material. Consequently, the above phenomena arise as a result of significant decrease of the anisotropy of stiffness properties of the plate material under small values of E_2/E_1 and the study of that is very difficult and requires very cumbersome numerical investigations. Therefore, in the present paper these investigations are not considered and those will be studied in the other papers of the authors.

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REFERENCES

Bazant, Z. P. (1974) Three-dimensional harmonic functions near termination or intersection of gradient singularity lines: a general numerical method. *International Journal of Engineering Science*, **12**, 221–243.

Borodachev, N. M. (1976) Contact problem for a stamp with a rectangular base. *PMM Journal of Applied Mathematics Mechanics* **40**, 505–512.

Celep, Z. (1988) Rectangular plates resting on tensionless elastic foundation. *Journal of Engineering Mechanics Division ASCE*, **114**(12), 2083–2092.

Gladwell, G. and Iyer, K. J. (1974) Unbonded contact between a circular plate and an elastic half-space. *Journal of Elasticity*, **4**(2), 115–130.

Gol'denreizer, A. L. (1958) On Reissner's theory of the bending of plates. *Izvestiga AN SSSR, OTN*, **4**, 102–109.

Li, Hui and Dempsey, J. P. (1988) Unbonded contact of a square plate on an elastic half-space or a Winkler foundation. *Journal of Applied Mechanics*, **55**, 430–436.

Hussian, M. A., Pu, S. L. and Sadowsky, M. A. (1968) Cavitation at the ends of an elliptic inclusion inside a plate under tension. *Journal of Applied Mechanics, Transactions ASME*, **35**(3), 505–509.

Kerr, A. D. (1964) Elastic and viscoelastic foundation models. *Journal of Applied Mechanics*, **31**, 491–498.

Khathlan, A. A. (1994) Large-deformation analysis of plates on unilateral elastic foundation. *Journal of Engineering Mechanics*, **120**(8), 1820–1827.

Kocatürk, T. (1995) Rectangular anisotropic (orthotropic) plates on a tensionless elastic foundation. *Mecha`s of Composite Materials* **31**(3), 378–386.

Kromm, A. (1953). Verallgeneinerte theorie der plattenstatik. *Ingenieur-Archiv* **21**, 266–286.

Kromm, A. (1955). Über die Randguer Krafte bei gestutzten Platten. *Zamm* **35**, 231–242.

Reddy, J. N. (1990) A review of refined theories of laminated composite plates. *Shock and Vibration Digest* **22**(3), 3–13.

Reissner, E. (1945) The effect of transverse shear deformation on the bending of elastic plates. *Journal of Applied Mechanics* **12**(1), A-69–A-77.

Rvachev, V. L. (1959) The pressure on an elastic half-space of a stamp with wedge-shaped planform. *PMM Journal of Applied Mathematics Mechanics* **23,** 229–233.

Timoshenko, S. P. and Woinowsky-Krieger, S. (1959) *Theory of Plates and Shells*, McGraw-Hill, New York.

Vlasov, V. Z. and Leontiev, N. N. (1966) *Beams, Plates and Shells on Elastic Foundations*. Israel Program for Scientific Translation, Jerusalem, Israel.

Weitsman, Y. (1969) On the unbonded contact between plates and elastic half-space. *Journal of Applied Mechanics, Transactions of ASME* **36**(2), 198–202.

Weitsman, Y. (1970) On foundations that react in compression only. *Journal of Applied Mechanics*, **37,** 1019–1030.